More on Spans

Spans are a frequently difficult concept. We will, therefore, spend just a little more time on them.

Definition:

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\} = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n, \quad a_1, a_2, \ldots, a_n \in \mathbb{R}\}$$

so ALL the linear combinations of the vectors in the set.

Basic properties:

- The span of a non-empty set in V is a subspace of V.
- The span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \in V$ is the smallest subspace in V that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

So: if we can write a set as a span, (completely, exactly, nothing missing, added, etc). As in, we've seen examples like

$$W = \left\{a_2x^2 + a_0,\ a_2, a_0 \in \mathbb{R}\right\},$$

which is a subspace of $\mathcal{P}_2$. It's also in the form of a span, it's the set of all elements of the form

$$a_2\left(x^2\right) + a_0\,(1)\,, \quad a_2, a_0 \in \mathbb{R}.$$

Having it writable as a span (EXACTLY writable as a span) proves it to be a subspace.

Another example:

$$\left\{\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix},\ x_1, x_2 \in \mathbb{R}\right\}$$

is a subspace of $\mathbb{R}^3$, it's also a span

$$a_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad a_1, a_2 \in \mathbb{R}.$$

Write it as a span and you have a subspace, vector space, etc.

Something harder:

$$\left\{(a_3 - a_2 + a_1)x^2 + (2a_2 - 3a_1)x + 4a_1,\ a_1, a_2, a_2 \in \mathbb{R}\right\}$$

is a subspace. Look at the form of those vectors:

$$\begin{aligned}
&(a_3 - a_2 + a_1)x^2 + (2a_2 - 3a_1)x + 4a_3 \\
&= a_3x^2 - a_2x^2 + a_1x^2 + 2a_2x - 3a_1x + 4a_3 \\
&= a_3(x^2 + 4) + a_2(-x^2 + x) + a_1(x^2 - 3x), \qquad a_1, a_2, a_3 \in \mathbb{R},
\end{aligned}$$

a span, so it's a subspace of $\mathcal{P}_2$. Done. A little easier than checking the properties, though there are cases where that is necessary.

Checking if you've spanned the ENTIRE space is particularly difficult.

Does the set

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

span $\mathbb{R}^3$, is it a SPANNING SET for $\mathbb{R}^3$? That means the set has to have ALL VECTORS IN $\mathbb{R}^3$ in its span, its set of linear combinations. Need to check

$$a_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which gives us the equations

$$\begin{aligned} a_1 - a_2 &= x \\ -a_1 + 2a_2 + a_3 &= y \\ 2a_1 - a_2 + a_3 &= z \end{aligned}$$

leading to $a_1 = x + a_2$, which leads to these equations:

$$\begin{aligned} a_1 &= x + a_2 \\ 2a_2 + a_3 &= y + x + a_2 \\ -a_2 + a_3 &= z - 2x - 2a_2 \end{aligned}$$

then the following two results for a_3:

$$\begin{aligned} a_3 &= y + x - a_2 \\ a_3 &= z - 2x - a_2 \end{aligned}$$

so we get two different values for a_3, a contradiction, unless

$$y + x - a2 = z - 2x - a2 \qquad \Longrightarrow \qquad 3x + y - z = 0.$$

Later we'll be able to be COMPLETELY certain about this sort of thing, but this basically shows us that we get an answer only if $3x + y - z = 0$, only on that plane.

Notice that:

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

This is a big deal. We need three directions to get the volume of $\mathbb{R}^3$, but one of our vectors is actually within the plane created by the other two. We need THREE DIFFERENT directions, we really only have two.

Linear Independence

... and dependence, as well. They are quite heavily connected. Consider the following spanning set for $\mathbb{R}^2$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

Do we really need all three of those terms to span the space? Any two of them will do (check, if you feel like it).

Look at the last two. They're in similar directions, but they are in DIFFERENT directions. As a result, a span of the two of them has two directions to work with and, as a result, covers a plane, in this case $\mathbb{R}^2$. The two vectors are in different directions because

$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad a \begin{bmatrix} 2 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{for all } a \in \mathbb{R}.$$

Problem is, that's a property of just two vectors. It's not as much a property of the whole set, and there are two different equations to check. Notice, however, that there's one commonality between those two $\neq$ sets:

$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} = b \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{means that } a = b = 0$$

which is the exact same as

$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{0} \qquad \text{means that} a = b = 0.$$

That statement is totally equivalent to the two earlier ones.

Now to bring back the full set of three.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{implies that} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{0}$$

so we have a linear combination of the three that adds to zero, but we didn't have to multiply them all by zero. The vectors going in different directions could only be linearly combined to $\mathbf{0}$ if they were all multiplied by zero. The set with an unnecessary term could add up to zero without resorting to multiplying by zero. It's not what you can make out of the vectors, it's how you can make it.

Official definition time:

Definition: a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is *Linearly Independent* if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_3 = \mathbf{0} \quad \text{implies that } a_1 = a_2 = \cdots = a_n = 0.$$

So, a linear combination of the vectors adding to zero has to be made using ONLY zero coefficients. Note that this linear combination is ALWAYS POSSIBLE. A set of vectors is linearly independent if that is the *only* option.

Definition: a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is *Linearly Dependent* if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_3 = \mathbf{0}$$

can be done without all a values equal to zero. Alternative Definition: if the set of vectors is not Linearly Independent. As before, the zero option is always an option. It's just that linearly dependent sets have other options.

This does tie in quite tightly with the concept of looking for ways to write one vector as the others. If $\mathbf{v}_1$ can be created out of the others:

$$\mathbf{v}_1 = b_2\mathbf{v}_2 + b_3\mathbf{v}_3 + \cdots + b_n\mathbf{v}_n$$

which converts easily to

$$\mathbf{0} = (-1)\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 + \cdots + b_n\mathbf{v}_n,$$

so linearly dependent. You can do this the other way too, it's harder.

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are linearly independent, then none of these are an option, NONE OF THE VECTORS can be written in terms of THE OTHERS.

Example: Here's two sets to consider linear dependence, etc,

$$A : \left\{x^2 + x, x^2 + 1, x + 1\right\} \qquad B : \left\{x^2 - x, x^2 + 1, x + 1\right\}$$

A: need to check the available answers for:

$$a_1(x^2 + x) + a_2(x^2 + 1) + a_3(x + 1) = 0$$

which converts to

$$(a_1 + a_2)x^2 + (a_1 + a_3)x + (a_2 + a_3) = 0,$$

$$a_1 + a_2 = 0, \qquad a_3 + a_1 = 0, \qquad a_2 + a_3 = 0$$

and, finally,

$$a_1 = -a_2, \qquad a_3 = -a_1 = a_2 \qquad a_3 = -a_2.$$

What does this say? Well, $a_3 = a_2$ but also $a_3 = -a_2$, only possible if they're both zero, which puts $a_1 = 0$ as well. ONLY THE ZERO ANSWER. Those three vectors are Linearly Independent, LI.

B: need to check the available answers for:

$$a_1(x^2 - x) + a_2(x^2 + 1) + a_3(x + 1) = 0$$

which converts to

$$(a_1 + a_2)x^2 + (-a_1 + a_3)x + (a_2 + a_3) = 0$$

$$a_1 + a_2 = 0, \qquad a_3 - a_1 = 0, \qquad a_2 + a_3 = 0$$

This leads to

$$a_1 = -a_2, \qquad a_3 = a_1 = -a_2, \qquad a_3 = -a_2$$

so we get multiple options, we can't break it down until everything is zero. We've got options. As a result, we have ourselves a Linearly Dependent set, LD.

Basic Properties

- The set $\{\mathbf{0}\} \in V$ is linearly dependent in V.

- Any set of vectors that includes $\mathbf{0}$ is linearly dependent in a vector space.

- The set $\{\mathbf{x}, \mathbf{y}\}$ is linearly dependent only if $a\mathbf{x} = \mathbf{y}$ OR $b\mathbf{y} = \mathbf{x}$.

These are fairly easy to show. Try it!

Example: $\cos(x)$ and $\sin(x)$ are linearly independent in $\mathcal{F}$.

Not has hard as it looks. We need to find all solutions to

$$a\cos(x) + b\sin(x) = 0, \qquad \text{for all } x \in \mathbb{R}.$$

Look at two points of x, $x = 0$ and $x = \frac{\pi}{2}$. They give us

$$a(1) + b(0) = 0, \qquad a(0) + b(1) = 0$$

which gives us $a = b = 0$, so Linearly Independent.

Example:

$$\left\{ \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \right\}$$

Again, check every single solution of the system

$$a_1 \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This is true when the following equations are true

$$3a_1 + a_2 + 2a_3 = 0, \qquad 2a_1 - a_2 + a_3 = 0 \qquad 2a_1 + 2a_2 + 2a_3 = 0 \qquad a_2 - a_3 = 0.$$

Notice the last one, gives us $a_2 = a_3$. Using that on the remaining three gives us

$$3a_1 + 3a_2 = 0, \qquad 2a_1 = 0, \qquad 2a_1 + 4a_2 = 0.$$

Using the first one we get $a_1 = -a_2$, using the second $a_1 = 0$, the last $a_1 = -2a_2$, meaning the only solution $a_1 = a_2 = a_3 = 0$.

A few exercises from the text, again:

Section 4.1
5.b)d)f), 6.b)d)

Section 4.2
1.a)b)c), 2.b)d)

Section 5.2
1.b)d)f)h), 2.b)d)f)h)